Maple Operators

In this worksheet a Maple operator means such procedures which assign functions to functions. The most frequent operator is the D differential operator concerning the differential equations. As a preparation, we are going to examine the relation of D with the @ composition operator and introduce the @@ operator used at the composition of the identical objects. Finally, we are going to show examples how to solve the differential equations with the Laplace transformation, the calculation of series and in a numerical way.

The composition of two functions is created by the @ operator. Thus the compound function of the h1 and h2 functions are the h1@h2. So far it seems easy. However, we have to be careful.

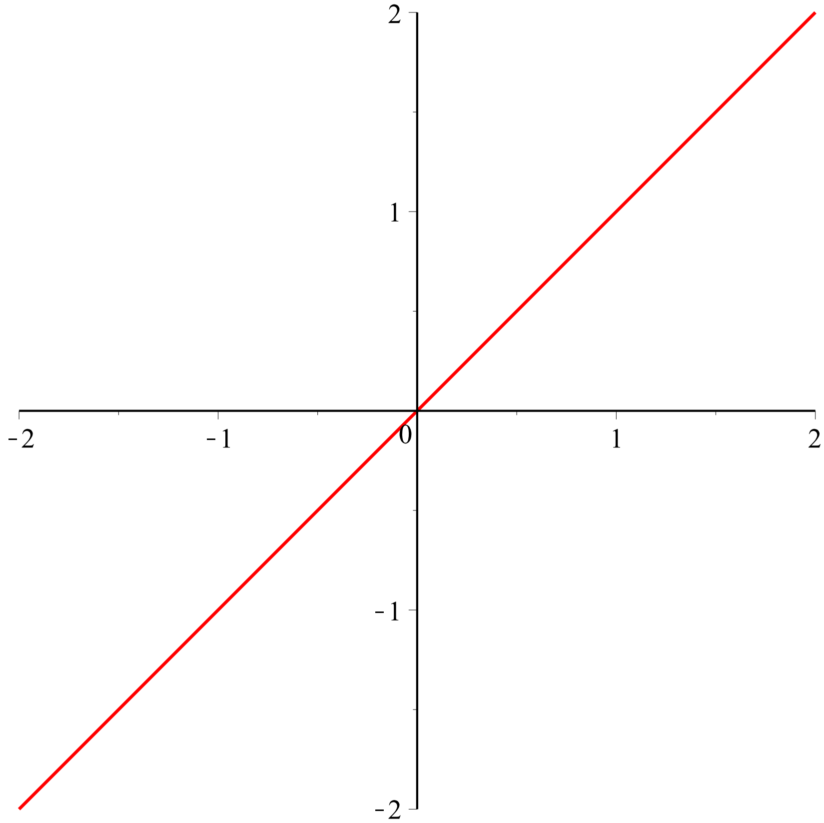
Let's look at the compound function considered in two orders of the f=sin and g=arcsin functions and plot the graph of the two compound functions. You may ask why. The sin and arcsin are the inverse of each other thus both compounds return the y=x axis. And a line is easy to imagine.

```
> restart; f := sin@arcsin; g := arcsin@sin
```

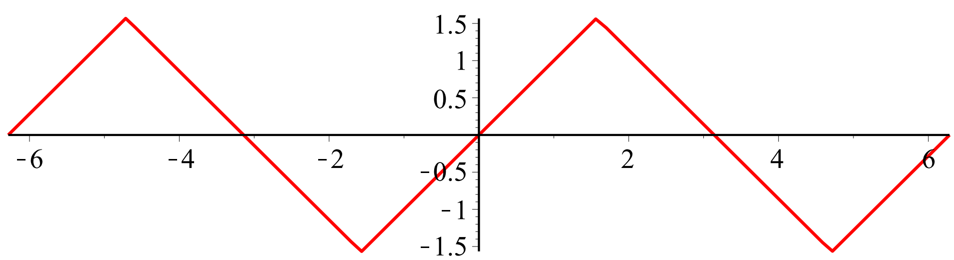
$$f := \sin@\arcsin$$

$$g := \arcsin@\sin \quad (1)$$

```
> plot(f, -2..2, scaling = constrained)
```



> *plot*(*g*, −2 Pi ..2 Pi, *scaling* = *constrained*)



We hope that you are also surprised **by** *this graph. We requested the graph of the f function* **in** *the* −2 ..2 *interval* **and** *the system did* **not** *recognize that the* arcsin (*the inner function*) *is interpreted only* **in** *the* [−1 ..1] *interval. So the graph is* **not** *correct.*

But why is the g function like saw teeth when it should be a straight line? The answer is that because *it does not have to be a line. The arcsin is the inverse of the sin function only in the* $\frac{-\mathrm{Pi}}{2} .. \frac{\mathrm{Pi}}{2}$

interval. So the usage of a computer-algebra system really does teach us mathematical strictness.

Differentiate the f and g compound functions with the D operator.

> $\mathrm{D}(f)$

$$\cos@\arcsin\left(z \rightarrow \frac{1}{\sqrt{1-z^2}}\right) \quad \mathbf{(2)}$$

> $\mathrm{D}(g)$

$$\left(z \rightarrow \frac{1}{\sqrt{1-z^2}}\right)@\sin \cos \quad \mathbf{(3)}$$

Good! The system expressed the derivatives with functions but the derivative of the arcsin does not have a name so we received that directly by the arrow notation. The usage of such nameless functions can be very useful when working with Maple.

How can we get the second order derivative of a function? There are two options. The first is that we apply the D operator to the D(f) differentiated function. The second is the usage of the @ operator.

> $f := 'f'$

$$f \quad \mathbf{(4)}$$

> $'\mathrm{D}(\mathrm{D}(f))' = \mathrm{D}(\mathrm{D}(f))$

$$\mathrm{D}(\mathrm{D}(f)) = \left(\mathrm{D}^{(2)}\right)(f) \quad \mathbf{(5)}$$

> $'(\mathrm{D}@\mathrm{D})(f)' = (\mathrm{D}@\mathrm{D})(f)$

$$(\mathrm{D}@\mathrm{D})(f) = \left(\mathrm{D}^{(2)}\right)(f) \quad \mathbf{(6)}$$

We can create higher order derivatives in this way but in the case of long compositions this method is rather cumbersome. Thus Maple provides the @@ operator to shorten the repeated composition.

> $'((\mathrm{D}@\mathrm{D})@\mathrm{D})(f)' = ((\mathrm{D}@\mathrm{D})@\mathrm{D})(f)$

$$((\mathrm{D}@\mathrm{D})@\mathrm{D})(f) = \left(\mathrm{D}^{(3)}\right)(f) \quad \mathbf{(7)}$$

> $(D@@3)(f) = \left(\mathrm{D}^{(3)}\right)(f)$

$$(D@@3)(f) = \left(\mathrm{D}^{(3)}\right)(f) \quad \mathbf{(8)}$$

> $'\mathrm{D}^{(0)}' = \mathrm{D}^{(0)}$

$$\mathrm{D}^{(0)} = (() \rightarrow args) \quad \mathbf{(9)}$$

> $\%(x)$

$$x = x \quad \mathbf{(10)}$$

So the D@@n denotes the n order operator differentiated. The last command shows that the D@@0 is the identical operator.

Naturally we can easily create an operator. The following procedure defines the linear difference operator of the f function which is used in the numerical analysis in many ways.

```
> Differencia:=proc(f)
  global h;local x;
  unapply(simplify(f(x+h)-f(x)),x);
  end;
```

$$\textbf{proc}(f)\ \textbf{local}\, x;\ \textbf{global}\, h;\ unapply(simplify(f(x+h)-f(x)),x)\ \textbf{end proc} \quad (11)$$

Notice that the Difference procedure behaves like the D operator. The following examples are obvious.

> $Differencia(f)$

$$x \rightarrow f(x+h) - f(x) \quad (12)$$

> $Differencia(\%)$

$$x \rightarrow f(x+2\,h) - 2\,f(x+h) + f(x) \quad (13)$$

> $(Differencia@Differencia)(f)$

$$x \rightarrow f(x+2\,h) - 2\,f(x+h) + f(x) \quad (14)$$

> $\left(\left(\mathrm{D}^{(3)}\right)(Differencia)\right)(f);$

$$\left(\left(\mathrm{D}^{(3)}\right)(Differencia)\right)(f) \quad (15)$$

> $Differencia\left(x \rightarrow x^2\right)$

$$x \rightarrow 2\,x\,h + h^2 \quad (16)$$

> $Differencia(\%)$

$$x \rightarrow 2\,h^2 \quad (17)$$

> $Differencia(\%)$

$$x \rightarrow 0 \quad (18)$$

After this preparation let's see the differential equations. So far we have used the diff procedure to give the differential equations but the D operator can be used as well. The (19) and the (20) describe the same differential equation.

> $\frac{\mathrm{d}^2}{\mathrm{d}x^2}\, y(x) + \frac{y(x)}{x^2} = 0$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\, y(x)\right) + \frac{y(x)}{x^2} = 0 \quad (19)$$

> $((\mathrm{D}@\mathrm{D})(y))(x) + \frac{y(x)}{x^2} = 0$

$$\left(\left(\mathrm{D}^{(2)}\right)(y)\right)(x) + \frac{y(x)}{x^2} = 0 \quad (20)$$

Solve both differential equations for the y(x) unknown function.

> $dsolve\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\, y(x)\right) + \frac{y(x)}{x^2} = 0, y(x)\right)$

$$y(x) = \_C1\,\sqrt{x}\,\sin\left(\frac{1}{2}\sqrt{3}\,\ln(x)\right) + \_C2\,\sqrt{x}\,\cos\left(\frac{1}{2}\sqrt{3}\,\ln(x)\right) \quad (21)$$

> $dsolve\left(\left(\left(D^{(2)}\right)(y)\right)(x)+\frac{y(x)}{x^2}=0, y(x)\right)$

$$y(x) = \_C1\,\sqrt{x}\,\sin\left(\frac{1}{2}\sqrt{3}\,\ln(x)\right)+\_C2\,\sqrt{x}\,\cos\left(\frac{1}{2}\sqrt{3}\,\ln(x)\right) \quad (22)$$

It can be seen that we get the same solution with both notations. It would be a problem if we had not received this.

We have got the general solution of the second-order differential equation in which the values of the _C1 and _C2 parameters can be chosen freely. We can make initial conditions about the y and its first derivative so that the solution should be obvious.

> $y(1)=1,\ (D(y))(1)=1$

$$y(1)=1,\ (D(y))(1)=1 \quad (23)$$

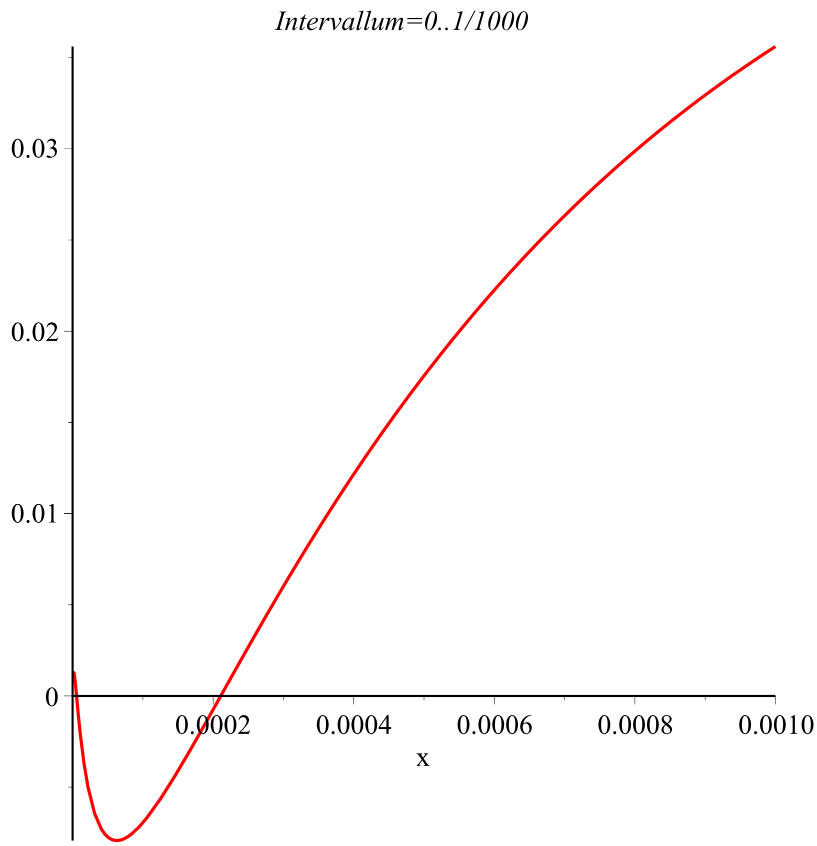
> $dsolve\left(\left\{\frac{d}{dx}\left(\frac{d}{dx}\,y(x)\right)+\frac{y(x)}{x^2}=0, y(1)=1,\ (D(y))(1)=1\right\}, y(x)\right)$

$$y(x)=\frac{1}{3}\sqrt{3}\,\sqrt{x}\,\sin\left(\frac{1}{2}\sqrt{3}\,\ln(x)\right)+\sqrt{x}\,\cos\left(\frac{1}{2}\sqrt{3}\,\ln(x)\right) \quad (24)$$

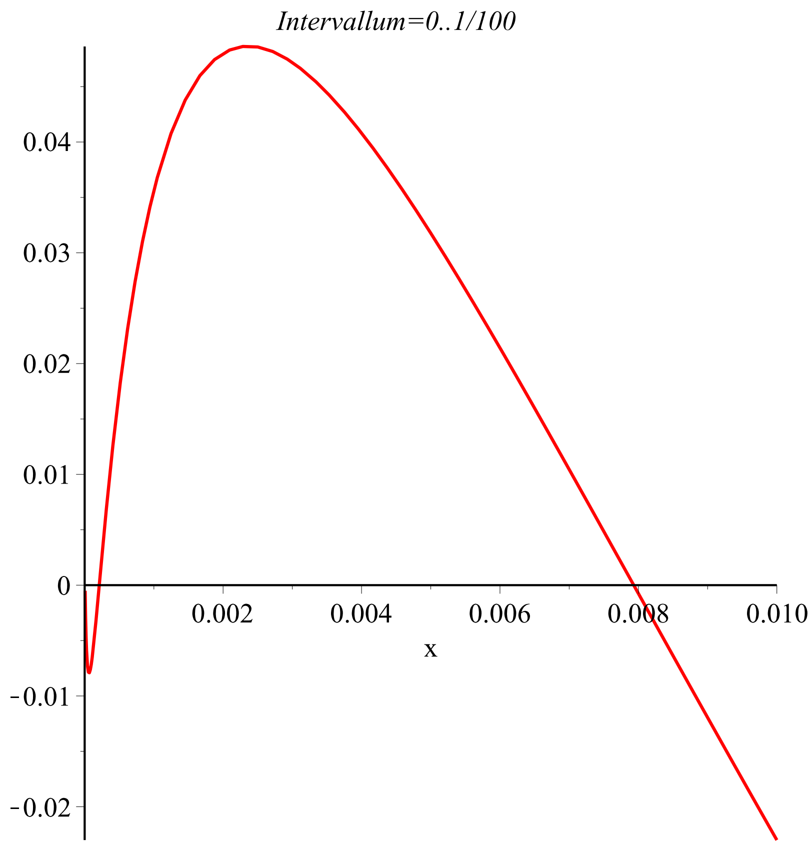
We have the solution which satisfies the (23) initial condition. The y(x) is a very interesting function. We are going to show only two graphs but we encourage you to examine this function further because it is worth it.

> $assign(\%)$

> $plot(y(x), x=0\,..0.001, title=Intervallum{=}0..1/1000)$



> $plot(y(x), x = 0 ..0.01, title = Intervallum{=}0..1/100)$



We are going to write the following linear second-order inhomogeneous differential equation with a constant coefficient by the D operator. Use the method=laplace option of the dsolve procedure in which case the system uses the Laplace transformation to solve the differential equation.

> $y := 'y';$

$$y \quad (25)$$

> $2\,\left(\left(D^{(2)}\right)(y)\right)(x) - 3\,(D(y))(x) + y(x) = \sin(x)$

$$2\,\left(\left(D^{(2)}\right)(y)\right)(x) - 3\,(D(y))(x) + y(x) = \sin(x) \quad (26)$$

> $dsolve\left(2\,\left(\left(D^{(2)}\right)(y)\right)(x) - 3\,(D(y))(x) + y(x) = \sin(x), y(x), method = laplace\right)$

$$y(x) = \frac{3}{10}\cos(x) - \frac{1}{10}\sin(x) + \frac{1}{2}\,(4\,(D(y))(0) - 2\,y(0) + 1)\,e^{x} + \frac{2}{5}\,e^{\left(\frac{1}{2}x\right)}\,(-5\,(D(y))(0) + 5\,y(0) - 2) \quad (27)$$

For the first try, we don't get the solution of the following second-order differential equation with a closed formula. Maple only repeats the input.

> $subs(y(0) = 2, (D(y))(0) = 3, rhs(\%))$

(28)

$$\frac{3}{10}\cos(x) - \frac{1}{10}\sin(x) + \frac{9}{2}e^{x} - \frac{14}{5}e^{\left(\frac{1}{2}x\right)} \qquad \textbf{(28)}$$

For the first try, we don't get the solution of the following second-order differential equation with a closed formula. Maple only repeats the input.

> $\frac{d^2}{dx^2}y(x) + \left(1 - \frac{1}{x^2+2}\right)y(x) = 0$

$$\frac{d}{dx}\left(\frac{d}{dx}y(x)\right) + \left(1 - \frac{1}{x^2+2}\right)y(x) = 0 \qquad \textbf{(29)}$$

> $dsolve(\textbf{(29)}, y(x))$

$$y(x) = DESol\left(\left\{\frac{d}{dx}\left(\frac{d}{dx}\_Y(x)\right) + \left(1 - \frac{1}{x^2+2}\right)\_Y(x)\right\}, \{\_Y(x)\}\right) \qquad \textbf{(30)}$$

However, with a little help we can get the system to work. Let's try to solve the differential equation above with the calculation of series after the initial conditions.

> $y(0) = 1, (D(y))(0) = 1$

$$y(0) = 1, (D(y))(0) = 1 \qquad \textbf{(31)}$$

> $Order := 10$

> $dsolve\left(\left\{y(0) = 1, (D(y))(0) = 1, \frac{d}{dx}\left(\frac{d}{dx}y(x)\right) + \left(1 - \frac{1}{x^2+2}\right)y(x) = 0\right\}, y(x), series\right)$

$$y(x) = \left(1 + x - \frac{1}{4}x^2 - \frac{1}{12}x^3 - \frac{1}{96}x^4 - \frac{1}{96}x^5 + O(x^6)\right) \qquad \textbf{(32)}$$

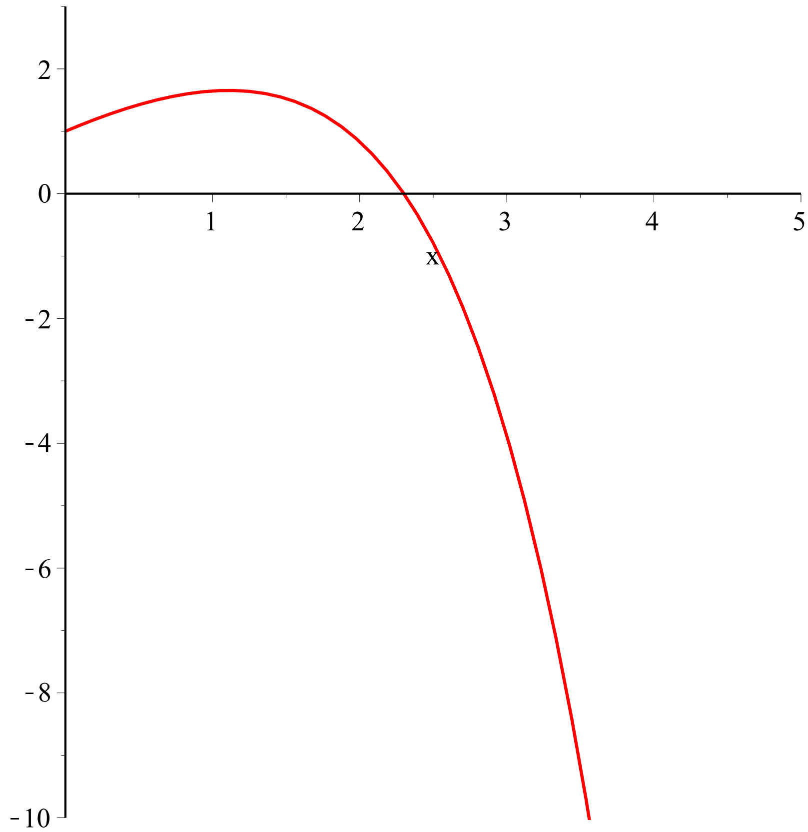
By having set the Order environment variable to 10 we want the coefficients of the series to be determined until the 10th exponent of the x in the power series. We indicate for the dsolve by the input of the series option that we want to get the solution with the calculation of series.

The power series has to be converted to a polynomial form for the sake of representation. It is done by the polynom option of the convert procedure.

> $convert(rhs(\%), polynom)$

$$1 + x - \frac{1}{4}x^2 - \frac{1}{12}x^3 - \frac{1}{96}x^4 - \frac{1}{96}x^5 \qquad \textbf{(33)}$$

> $plot\left(1 + x - \frac{1}{4}x^2 - \frac{1}{12}x^3 - \frac{1}{96}x^4 - \frac{1}{96}x^5, x = 0..5, -10..3\right)$



Naturally this graph gives the approximation of the solution and it does this precisely only near the x=0. By approaching with higher degree polynomials we can get a better approximation in a bigger interval.

Let's try to solve the same differential equation in a numerical way. We only have to give the numeric option for this. We get the solution as the list of procedures if we use the output=listprocedure option.

> $mo := dsolve\left(\left\{y(0)=1,\ (D(y))(0)=1,\ \frac{d}{dx}\left(\frac{d}{dx}\,y(x)\right)+\left(1-\frac{1}{x^2+2}\right)y(x)=0\right\}, y(x), numeric, output=listprocedure\right)$

$$\left[x=\mathbf{proc}(x)\ \ldots\ \mathbf{end\ proc},\ y(x)=\mathbf{proc}(x)\ \ldots\ \mathbf{end\ proc},\ \frac{d}{dx}\,y(x)=\mathbf{proc}(x)\ \ldots\ \mathbf{end\ proc}\right] \tag{34}$$

Since this is a second-order differential equation we get a procedure for the x,y(x) and the D(y)(x) as a solution. We want to have the y(x) procedure plotted thus it is useful to break up the procedures of the list.

> $X := subs(mo, x)$

$$X := \mathbf{proc}(x)\ \ldots\ \mathbf{end\ proc} \tag{35}$$

> $Y := subs(mo, y(x))$

$$Y := \textbf{proc}(x)\ \dots\ \textbf{end proc} \tag{36}$$

> $DY := subs\left(mo, \frac{d}{dx}\, y(x)\right)$

$$DY := \textbf{proc}(x)\ \dots\ \textbf{end proc} \tag{37}$$

These procedures give the solution and its derivative at the given x spot. Let's calculate their values at the x=1 spot.

> $'Y(1)' = Y(1)$

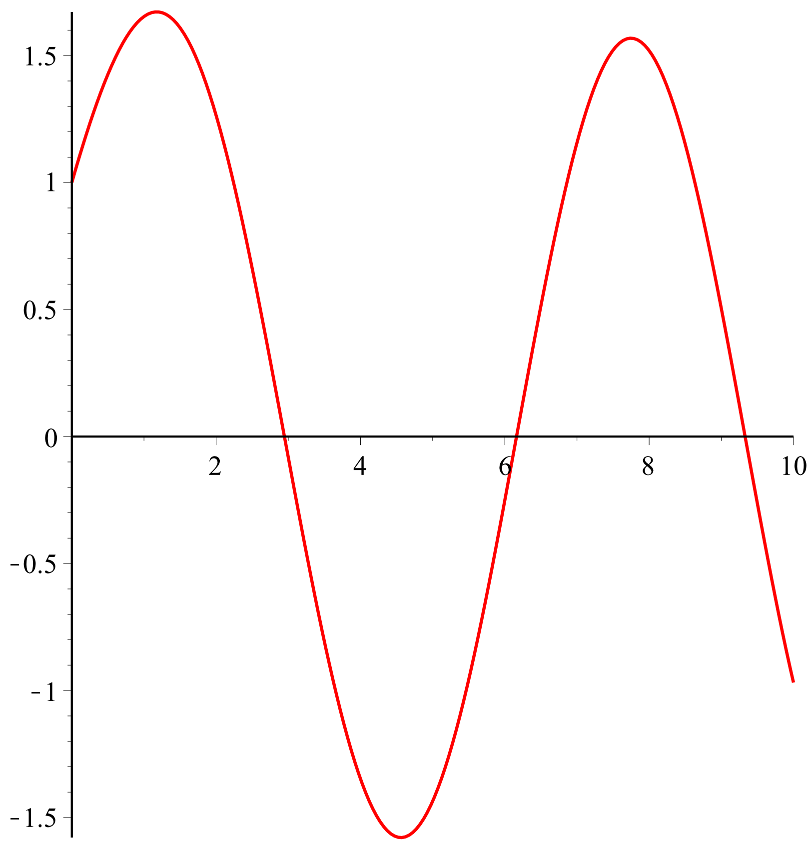
$$Y(1) = 1.65377868037883946 \tag{38}$$

> $'(D(Y))(1)' = DY(1)$

$$(D(Y))(1) = 0.203523011254604341 \tag{39}$$

Finally, let's plot the procedures received for the X and Y. In this case each of the X and Y is a procedure. We are going to use the parametric syntax of the plot to draw it.

> $plot([X, Y, 0..10], numpoints = 200)$



>

The syntax of the solution is totally different from the solution we received with the calculation of series.

Which graph should we accept? Notice that in the small area of the x=0 (approximately in the [0..2] interval) the kind of the two solutions does not differ from each other. But in the case of larger x-es we cannot trust the polynomial approximation. Anyway, we can presume based on the numerical solution that the solution of the differential equation is limited and oscillates in the [képlet] interval.

What Have You Learnt About Maple?

1. The @@ denotes the n order compound operator. If the n=0 then we get the identical operator.

2. We can convert a sub sum of the power series to a polynomial form with the convert(series,polynom) instruction.

3. We can get the solution of the differential equation with different methods, with the following options of the dsolve procedure:

- laplace – Laplace transformation
- series – power calculation of series
- numeric – in a numerical way

Exercises

1. Solve the following differential equations with the help of the Laplace transformation.

a) $\frac{d^2}{dx^2} y(x) + y(x) = \tan(x);$

b) $\frac{d^2}{dx^2} y(x) - 6 \cdot y(x) + 13 \cdot y(x) = \exp(x) \cdot cos(x)$

c)

$\frac{d^2}{dx^2} y(x) - y(x) - 6 \cdot y(x) = 6 \cdot \exp(3 \cdot x) + 2 \cdot exp(-2 \cdot x),$ y(0)=0, D(y)(0)=$\frac{4}{5}$

d) $\frac{d}{dt} x(t) = -7x + y + 5, \quad \frac{d}{dt} y(t) = -2x - 5y - 37, \quad x(0) = 0,\ y(0) = 0$

□

2. Solve the following differential equations with the help of the calculation of series.!

a) $(1-x)^2 \left(\frac{d^2}{dx^2} y(x)\right) - 2\,(1-x)\left(\frac{d}{dx} y(x)\right) - 2\,y(x) = 0, \quad y(0) = 1, \quad D(y)(0) = 0$

b) $x^2 \left(\frac{d^2}{dx^2} y(x)\right) + x\left(\frac{d}{dx} y(x)\right) + (x^2 - 1) \cdot y(x) = 0,$ y(0)=0, $(D(y))(0) = \frac{1}{2}$

c)

$$\left(\frac{d^2}{dx^2}\, y(x)\right) - x\left(\frac{d}{dx}\, y(x)\right) - 2\cdot y(x) = 0$$

d) $\left(\frac{d^2}{dx^2}\, y(x)\right) + x^2\, y(x) = 0$